Robust Control Design with Real-Parameter Uncertainty and Unmodeled Dynamics

Hsi-Han Yeh,* Siva S. Banda,† and Sharon A. Heise‡
Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio 45433
and
Andrew C. Bartlett§
University of Massachusetts—Amherst, Amherst, Massachusetts 01003

This paper presents a design methodology for the synthesis of a robust controller that accounts for both unmodeled dynamics and structured real-parameter uncertainty for multiple-input/multiple-output systems. The unmodeled dynamics are assumed to be characterized as a single block dynamic uncertainty at a point in the closed-loop system. In a design aimed at constraining both the H_∞ norm of a certain disturbance transfer matrix and a quadratic Gaussian performance index under their respective bounds, a surrogate system may be formed by modeling the structured real-parameter uncertainty as additional noise inputs and additional weights at the existing noise inputs and measurement outputs of the system. Application of a Riccati equation approach to this surrogate system then yields a robust controller that, when used in the actual system, will result in a closed-loop system that has the same H_∞ bound and quadratic Gaussian performance index bound as the surrogate system, even in the presence of given real-parameter variations.

Introduction

ROBUST design of multivariable feedback systems with unmodeled dynamics may be achieved by minimizing the H_∞ norm (or the greatest singular value maximized over the entire frequency spectrum) of the return difference of the loop-transfer matrix, or the sensitivity matrix, depending on how the unmodeled dynamics are represented as plant perturbations. When the parameters of the nominal plant model are fixed, the H_∞ optimization may be carried out by methods given in such papers as Doyle et al.,[1] Chang et al.,[2] and Safonov et al.[3] In most advanced aerospace control systems, the plant uncertainties not only exist in the form of unmodeled dynamics, such as high-frequency flexible modes due to aeroservoelasticity or the truncation of actuator and sensor dynamics, but also in the form of data inaccuracy, such as the approximate location of the center of gravity of an aircraft, imprecise knowledge of mass and moments of inertia, and uncertainty in aerodynamic coefficients. The latter form appears as the structured uncertainty in the real parameters of the mathematical model of an aircraft.

Most existing robust control design methods deal with one form of uncertainty or another. For example, H_∞ optimization methods as presented in Refs. 1–3 yield control systems that are robust against unmodeled dynamics. Kharitonov's theorem[4] and its extensions[5] are an effective tool in dealing with the real-parameter plant uncertainty. However, when both unmodeled dynamics and structured real-parameter uncertainties are present, the design problem becomes even more difficult. It is possible to treat real-parameter uncertainties as magnitudes of frequency-transfer functions and to use the structured singular-value method[6] for the synthesis. It is desirable, however, to develop an H_∞ design technique that addresses real-parameter uncertainties without modeling them as complex variations.

There are relatively few papers[7–11] addressing the problem of robust control in the presence of both unmodeled dynamics and real-parameter uncertainties. References 7–9 deal with the analysis aspects and Refs. 10 and 11 address the design aspects of the problem. While the approach of Ref. 10 is highly graphical and does not appear to be amenable to the use of computers for multiple-input/multiple-output systems, the work of Ref. 11 is applicable to only single-input/single-output systems.

In a recent paper, Bernstein and Haddad[12] use the guaranteed-cost approach of Chang and Peng,[13] in conjunction with the quadratic Lyapunov bound recently suggested by Peterson and Hollot,[14] and formulate a modified Riccati equation whose solutions guarantee the robust stability and quadratic-Gaussian performance (an H_2 norm) of the closed-loop system over a range of structured real-parameter uncertainties. Since the Peterson-Hollot bound is differentiable with respect to the solution of the Riccati equation, the application of an optimal projection approach to the modified Riccati equation then yields linear-quadratic-Gaussian (LQG) compensator that tolerates structured real-parameter uncertainties. In a later publication, Bernstein and Haddad[15] extend the results of Petersen[16] and Khargonekar et al.[17] to develop a design methodology for fixed-order, H_2 sub-optimal control (where an upper bound of the H_2 performance criterion is minimized), which includes as a design constraint a bound on H_∞ disturbance attenuation. Despite the similarity in the Riccati equations involved in both Refs. 12 and 15, the results of Ref. 12 do not guarantee any H_∞ bound, whereas the results of Ref. 15 do not tolerate real-parameter uncertainties.

The purpose of this paper is to extend the techniques developed in Refs. 12 and 15 to yield a design methodology for robust control in the presence of both unmodeled dynamics and structured real-parameter uncertainties. A basic concept contributed in this paper is that, in the H_∞/H_2 robust design, the effect of structured real-parameter uncertainty may be modeled as additional noise inputs and additional weights at the existing noise inputs and measurement outputs of the system. The H_∞ design, subject to a minimal H_2

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*Electronics Engineer, Control Dynamics Branch.

†Aerospace Engineer, Control Dynamics Branch. Associate Fellow AIAA.

‡Captain, U.S. Air Force, Stability and Control Engineer, Control Dynamics Branch. Member AIAA.

§Graduate Student, Department of Electrical Engineering.

performance bound and structured real-parameter uncertainties, can thus be carried out on a surrogate system with fixed parameters and added noise and output intensity. The resulting control system is conservative since the additional noise inputs and the additional weights are not equivalent to the given real-parameter uncertainties.

Problem Formulation

H-Infinity Constrained LQG Control with Real-Parameter Uncertainty

Let the nth-order stabilizable and detectable plant with structured real-parameter uncertainties be given by (see Fig. 1)

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) + D_1 w_1(t) \tag{1}$$

$$y(t) = (C + \Delta C)x(t) + D_2 w_2(t) \tag{2}$$

$$z_{1\infty}(t) = (E_{1\infty} + \Delta E_{1\infty})x(t) \tag{3}$$

$$z_{2\infty}(t) = (E_{2\infty} + \Delta E_{2\infty})u(t) \tag{4}$$

where x is the n-dimensional state vector, u is an m-dimensional control vector, y is an r-dimensional output vector, w_1 and w_2 are p_1- and p_2-dimensional uncorrelated white noise vectors, respectively, $z_{1\infty}$ and $z_{2\infty}$ are $q_{1\infty}$- and $q_{2\infty}$-dimensional measured outputs, respectively, ΔA, ΔB, ΔC, $\Delta E_{1\infty}$, and $\Delta E_{2\infty}$ are uncertain perturbations in the real parameters, D_1 and D_2 are noise parameters, and $E_{1\infty}$ and $E_{2\infty}$ are weighting matrices for the noise-transfer matrix. Note that since $E_{1\infty}$, and $E_{2\infty}$ are design parameters, not actual sensor parameters, they need not have independent perturbations. In design applications, $E_{1\infty}$ and $E_{2\infty}$ may be chosen to be dependent on the plant parameters, therefore they may have uncertain perturbations that are dependent upon ΔA, ΔB, and ΔC. The ∞ subscripts denote parameters and signals associated with the transfer-function matrix that will be subjected to the H_∞ norm constraint. $K(s)$ is the transfer matrix of the nth-order dynamic compensator, with state equations given by

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \tag{5}$$

$$u(t) = C_c x_c(t) \tag{6}$$

The state equation of the closed-loop system S_Δ may be written as

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta\tilde{A})\tilde{x}(t) + \tilde{D}\tilde{w}(t) \tag{7}$$

where

$$\tilde{x}(t) = [x(t)\ \ x_c(t)]^T; \qquad \tilde{w}(t) = [w_1(t)\ \ w_2(t)]^T \tag{8}$$

$$\tilde{A} \equiv \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}; \qquad \tilde{D} \equiv \begin{bmatrix} D_1 & 0 \\ 0 & B_c D_2 \end{bmatrix}$$

$$\Delta\tilde{A} \equiv \begin{bmatrix} \Delta A & \Delta B C_c \\ B_c \Delta C & 0 \end{bmatrix} \tag{9}$$

The disturbance transfer-function matrix $H(s,\Delta\tilde{A})$ from $\tilde{w}$ to $\tilde{z}$ ($\equiv [z_{1\infty}\ \ z_{2\infty}]^T$) may be written as

$$H(s,\Delta\tilde{A}) = (\tilde{E}_\infty + \Delta\tilde{E}_\infty)(sI - \tilde{A} - \Delta\tilde{A})^{-1}\tilde{D} \tag{10}$$

where

$$\tilde{E}_\infty \equiv \begin{bmatrix} E_{1\infty} & 0 \\ 0 & E_{2\infty}C_c \end{bmatrix}; \qquad \Delta\tilde{E}_\infty \equiv \begin{bmatrix} \Delta E_{1\infty} & 0 \\ 0 & \Delta E_{2\infty}C_c \end{bmatrix} \tag{11}$$

The robust synthesis problem considered in this paper is to determine the controller parameters (A_c, B_c, C_c) such that the

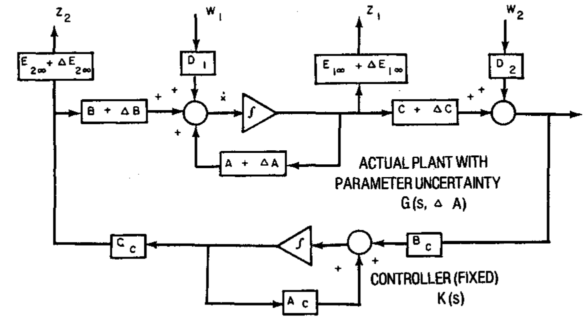

Fig. 1 Actual closed-loop system S_Δ.

following design criteria are satisfied over the entire range of $(\Delta A, \Delta B, \Delta C)$:

1) The closed-loop system S_Δ of Fig. 1 is asymptotically stable.

2) The disturbance transfer matrix $H(s,\Delta\tilde{A})$ from noise inputs w_1 and w_2 to measurement outputs $z_{1\infty}$ and $z_{2\infty}$ satisfies the constraint

$$\|H(s,\Delta\tilde{A})\|_\infty \le \gamma \tag{12}$$

for some prescribed positive constant γ.

3) The performance criterion

$$J(\Delta\tilde{A}) \equiv \lim_{t\to\infty} E[x^T(t)R_1 x(t) + u^T(t)R_2 u(t)] \tag{13}$$

is minimized. In Eq. (13), E is an expectation operator and R_1 and R_2 are weighting matrices that are positive semidefinite and positive definite, respectively.

Robustness in the Presence of both Unmodeled Dynamics and Real-Parameter Uncertainty

The robustness of a multivariable control system[18,19] may be measured by the maximum singular value of the sensitivity function or the complementary sensitivity function (the input-output transfer function), depending on how the unmodeled dynamics are represented as plant perturbations. Consider systems modeled with multiplicative perturbations as in Fig. 2, where

$L(s)$ = unmodeled and unstructured high-frequency dynamics

$G(s,\Delta\tilde{A})$ = plant model with real-parameter uncertainty

If $L(s) = 0$ and the system is stable for all $\Delta\tilde{A}$, then the system is stable for all $\Delta\tilde{A}$ and all $L(s)$ if

$$\max_\omega \bar{\sigma}[L(j(\omega))] < \frac{1}{\max_\omega \bar{\sigma}[GK(I + GK)^{-1}]} \tag{14}$$

where $\bar{\sigma}$ denotes the maximum singular value of its argument. However, $L(s)$ is usually unknown. Therefore, the problem is to design a controller $K(s)$ that minimizes the H_∞ norm of $H(s,\Delta\tilde{A})$ of Fig. 3 for all $\Delta\tilde{A}$, where

$$H(s,\Delta\tilde{A}) = G(s,\Delta\tilde{A})K(s)[I + G(s,\Delta\tilde{A})K(s)]^{-1} \tag{15}$$

With such a controller, the closed-loop system will be robustly stable in the presence of both real-parameter uncertainties and unmodeled dynamics described by $[I + L(s)]G(s,\Delta\tilde{A})$. It can be easily verified that by letting $D_2 = 0$, $E_{1\infty} = \Delta E_{1\infty} = 0$, $D_1 = B + \Delta B$, $E_{2\infty} = I$, and $\Delta E_{2\infty} = 0$ (see Fig. 1); this formulation deals with the class of unmodeled uncertainty that can be described by $G(s,\Delta\tilde{A})[I + L(s)]$. Furthermore, if one lets $D_2 = 0$, $E_{2\infty} = \Delta E_{2\infty} = 0$, $D_1 = B + \Delta B$, $E_{1\infty} = C$, and $\Delta E_{2\infty} = \Delta C$, this formulation then deals with the class of

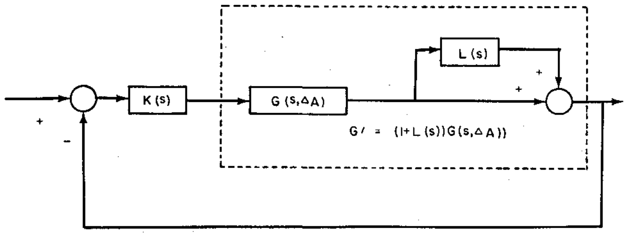

Fig. 2 Multiplicative representation of unmodeled dynamics.

unmodeled uncertainty that can be described by $G(s,\Delta\tilde{A}) + \Delta G(s,\Delta\tilde{A})$.

Usually some assumption on the unmodeled dynamics $L(s)$ can be made. In that case, the singular value function of the loop-transfer matrix $\bar{\sigma}[G(j\omega,\Delta\tilde{A})K(j\omega)]$ must be shaped in accordance with $L(j\omega)$ for maximum robustness with minimum conservatism.[19] In robust design with unmodeled dynamics only, a widely accepted loop-shaping technique is the LQG/LTR (loop transfer recovery).[19,20] In the present paper, the controller is LQG-optimal and H_∞-constrained at the same time. The LQG optimality offers some sort of loop shaping. How to select the quadratic weights of Eq. (13) to obtain a desired loop shape is left as a topic for further research. The present paper thus will consider the problem of constraining the H_∞ norm of the transfer function of Eq. (15) in the presence of real-parameter uncertainties without loop shaping, except for an LQG optimization as in Eq. (13).

For the transfer function of Eq. (15) to be the same as that in Eqs. (10) and (12), one may set

$$D_1 = 0; \quad D_2 = I; \quad E_{1\infty} = C; \quad E_{2\infty} = 0 \tag{16}$$

which gives

$$\Delta E_{1\infty} = \Delta C; \qquad \Delta E_{2\infty} = 0 \tag{17}$$

In the ensuing development, the theory will be developed for the general case, while an application will be shown for the robust control problem characterized by Eqs. (16) and (17).

Conditions for Simultaneous H_2 and H_∞ Bound

A lemma of Bernstein and Haddad[15] will be restated here as a starting point for the development of the disturbance-equivalence concept. First, the definition of some notations in the LQG theory is in order. For the moment, consider the case where there is no real-parameter uncertainty: $\Delta\tilde{A} = 0$. Substituting Eq. (6) into Eq. (13) and using the augmented state vector, one may rewrite the quadratic performance criterion as

$$J = \lim_{t\to\infty} E[\tilde{x}^T(t)R\tilde{x}(t)] = \mathrm{tr}Q_0 R \tag{18}$$

where R is the augmented weighting matrix of the closed-loop system, and Q_0 is the second moment or the covariance matrix of the augmented state vector of the closed-loop system:

$$R \equiv \begin{bmatrix} R_1 & 0 \\ 0 & C_c^T R_2 C_c \end{bmatrix}; \qquad Q_0 \equiv \lim_{t\to\infty} E[\tilde{x}(t)\tilde{x}^T(t)] \tag{19}$$

and Q_0 satisfies the Lyapunov equation

$$0 \equiv \tilde{A}Q_0 + Q_0\tilde{A}^T + V \tag{20}$$

where V is the power spectral density of the white Gaussian

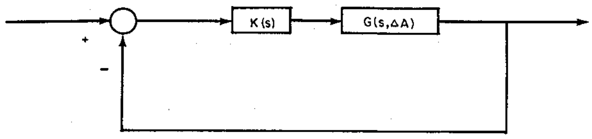

Fig. 3 Real-parameter uncertainty without unmodeled dynamics.

noise $\tilde{D}\tilde{w}$ of Eq. (7):

$$V \equiv \begin{bmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^T \end{bmatrix}; \qquad V_1 \equiv D_1 D_1^T; \qquad V_2 \equiv D_2 D_2^T \tag{21}$$

Define

$$R_\infty \equiv \begin{bmatrix} R_{1\infty} & 0 \\ 0 & C_c^T R_{2\infty} C_c \end{bmatrix}; \quad R_{1\infty} \equiv E_{1\infty}^T E_{2\infty}; \; R_{2\infty} \equiv E_{2\infty}^T E_{2\infty} \tag{22}$$

Lemma 1[15]: Let (A_c,B_c,C_c) be given such that there exists a non-negative-definite matrix Q_1 satisfying

$$0 = \tilde{A}Q_1 + Q_1\tilde{A}^T + \gamma^{-2}Q_1 R_\infty Q_1 + V \tag{23}$$

Then

$$(\tilde{A},[\gamma^{-2}Q_1 R_\infty Q_1 + V]^{1/2}) \tag{24}$$

is stabilizable if and only if

$$\tilde{A} \text{ is asymptotically stable} \tag{25}$$

The stability of $\tilde{A}$ implies that

$$\|H(s)\|_\infty \le \gamma \tag{26}$$

where $H(s)$ is determined by R_∞ and V in the same way as in Eqs. (10), (21), and (22) with $\Delta\tilde{A}$ and $\Delta\tilde{E}_\infty$ both equal to zero, and

$$Q_0 \le Q_1 \tag{27}$$

Consequently,

$$J \le \mathrm{tr}Q_1 R \tag{28}$$

Note that if the solution Q_1 is positive definite, then Eq. (24) is satisfied. Also, if V is positive definite or if $(\tilde{A},\tilde{D})$ is controllable, then Eq. (24) is satisfied. In applications, it is always simpler to check the stability of $\tilde{A}$ directly instead of checking Eq. (24).

Now let real-parameter variations be present. The following lemma establishes a condition under which the closed-loop system is asymptotically stable for all permissible uncertain perturbations in the real parameters.

Lemma 2: Let (A_c,B_c,C_c) be given such that there exists a positive-definite matrix Q satisfying

$$0 = \tilde{A}Q + Q\tilde{A}^T + \gamma^{-2}QR_\infty Q + \Omega + V \tag{29}$$

where Ω is symmetric and satisfies

$$\Omega \le \Delta\tilde{A}Q + Q\,\Delta\tilde{A}^T + \gamma^{-2}Q\,\Delta R_\infty Q \tag{30}$$

for all admissible $\Delta\tilde{A}$, where ΔR_∞ is the uncertain perturbation in R_∞, which may be dependent on $\tilde{A}$. Then there exists a non-negative-definite symmetric matrix Q_Δ satisfying

$$0=(\tilde{A}+\Delta\tilde{A})Q_\Delta+Q_\Delta(\tilde{A}+\Delta\tilde{A})^T+\gamma^{-2}Q_\Delta(R_\infty+\Delta R_\infty)Q_\Delta+V \tag{31}$$

and

$$(\tilde{A}+\Delta\tilde{A}) \quad \text{is asymptotically stable} \tag{32}$$

for all permissible $\Delta\tilde{A}$ that satisfy Eq. (30).

Proof: Define

$$\Psi\equiv\Omega-(\Delta\tilde{A}Q+Q\,\Delta\tilde{A}^T+\gamma^{-2}Q\,\Delta R_\infty Q)\le 0 \tag{33}$$

Then

$$\Omega\equiv\Psi+(\Delta\tilde{A}Q+Q\,\Delta\tilde{A}^T+\gamma^{-2}Q\,\Delta R_\infty Q) \tag{34}$$

Substituting Eq. (34) into Eq. (29) gives

$$0=(\tilde{A}+\Delta\tilde{A})Q+Q(\tilde{A}+\Delta\tilde{A})^T+\gamma^{-2}Q(R_\infty+\Delta R_\infty)Q+\Psi+V \tag{35}$$

Since a positive-definite solution exists for Eq. (35),

$$\{\tilde{A}+\Delta\tilde{A},[\gamma^{-2}Q(R_\infty+\Delta R_\infty)Q+\Psi+V]^{1/2}\} \tag{36}$$

is controllable. If a system is controllable, it is stabilizable. Therefore, by virtue of lemma 1, $\tilde{A}+\Delta\tilde{A}$ is asymptotically stable, and Eq. (31) is equivalent to

$$Q_\Delta=\int_0^\infty \varepsilon^{(\tilde{A}+\Delta\tilde{A})t}[\gamma^{-2}Q_\Delta(R_\infty+\Delta R_\infty)Q_\Delta+V]\times\{\varepsilon^{(\tilde{A}+\Delta\tilde{A})t}\}^T\,\mathrm{d}t\le 0 \tag{37}$$

This completes the proof of lemma 2.

Note that if V is positive definite, then, by virtue of Eq. (37), Q_Δ is positive definite. Furthermore, the assumption of lemma 2 may be relaxed to Q being non-negative definite. In that case, the asymptotic stability of $\tilde{A}+\Delta\tilde{A}$ may be concluded from Eq. (36), which is a necessary and sufficient condition. For all practical purposes, one would check the stability of $\tilde{A}+\Delta\tilde{A}$ directly, instead of checking Eq. (36). In many cases, this stability check may be performed by a procedure[21,22] involving a sufficient condition derived from the Kharitonov theorem.

Theorem 1: If the controller of the system of Fig. 1 satisfies the assumptions of lemma 2, then, for all permissible $\Delta\tilde{A}$, 1) the closed-loop system is asymptotically stable, and 2) the disturbance transfer matrix $H(s,\Delta\tilde{A})$ from noise inputs w_1 and w_2 to measurement outputs $z_{1\infty}$ and $z_{2\infty}$ satisfies the constraint

$$\|H(s,\Delta\tilde{A})\|_\infty\le\gamma \tag{38}$$

3) The performance criterion of Eq. (13) satisfies

$$J(\Delta\tilde{A})\equiv\lim_{t\to\infty}E[x^T(t)R_1x(t)+u^T(t)R_2u(t)]\le \mathrm{tr}QR \tag{39}$$

Proof: The stability of the closed-loop system for all permissible $\Delta\tilde{A}$ follows from lemma 2. The H_∞ norm constraint follows from the existence of a non-negative-definite solution of Eq. (31), the stability of $(\tilde{A}+\Delta\tilde{A})$, and the application of lemma 1. To prove the boundedness of the quadratic performance index, let Q_2 be the covariance matrix of the augmented state vector $\tilde{x}$ when the plant is subjected to uncertain real-parameter perturbations. Then Q_2 satisfies

$$0=(\tilde{A}+\Delta\tilde{A})Q_2+Q_2(\tilde{A}+\Delta\tilde{A})^T+V \tag{40}$$

and by definition,

$$J(\Delta\tilde{A})=\mathrm{tr}Q_2R \tag{41}$$

Subtracting Eq. (40) from Eq. (35) gives

$$0=(\tilde{A}+\Delta\tilde{A})(Q-Q_2)+(Q-Q_2)(\tilde{A}+\Delta\tilde{A})^T+\gamma^{-2}Q(R_\infty+\Delta R_\infty)Q+\Psi \tag{42}$$

Since $\tilde{A}+\Delta\tilde{A}$ is asymptotically stable, Eq. (42) is equivalent to

$$Q-Q_2=\int_0^\infty \varepsilon^{(\tilde{A}+\Delta\tilde{A})t}[\gamma^{-2}Q(R_\infty+\Delta R_\infty)Q+\Psi]\times\{\varepsilon^{(\tilde{A}+\Delta\tilde{A})t}\}^T\,\mathrm{d}t \tag{43}$$

Since Ψ is non-negative definite, we have

$$\Omega\le Q_2 \tag{44}$$

and consequently

$$J(\Delta\tilde{A})=\mathrm{tr}Q_2R\le\mathrm{tr}QR \tag{45}$$

This completes the proof of theorem 1.

Lemma 2 and theorem 1 have brought forth a notion that, in the H_∞/H_2 design, the real-parameter uncertainty may be modeled as additional weights at the noise inputs and measurement outputs. Let Ω be a constant matrix satisfying Eq. (30) and define a surrogate system S_0 to be the nominal system of Fig. 1 without any parameter uncertainty and with the white Gaussian noise intensity increased from V to $V+\Omega$. The actual system S_Δ of Fig. 1 has all its real-parameter uncertainties and has noise intensity V. If a controller $K(s)$ (whose coefficient matrices are A_c, B_c, and C_c) of S_0 causes the worst-case disturbance-to-output gain (when the Gaussian noises are replaced by deterministic disturbances) to be no greater than γ and the quadratic Gaussian performance index to be no greater than trQR, then the same controller, if used in S_Δ, also causes the disturbance-to-output gain and the quadratic Gaussian performance index of system S_Δ to have the same upper bounds as those of system S_0. With this notion, one only has to carry out the controller design for the nominal system S_0 in order to account for the real-parameter uncertainty in S_Δ.

The preceding paragraph is only a rudimentary account of the notion of weight substitution for real-parameter uncertainty in a robust design. Further development of this notion is possible after the explicit expressions of the bound Ω of the real-parameter uncertainty is derived in the next section.

Bounds on the Real-Parameter Uncertainty

Parameterization of the Real-Parameter Uncertainty

Assume $\Delta B=0$. The dual case of $\Delta C=0$ can be similarly treated. Assume that ΔA and ΔC can be represented by

$$\Delta A=\sum_1^p D_iM_iN_iE_i;\qquad \Delta C=\sum_1^p F_iM_iN_iE_i \tag{46}$$

where, for $i=1,2,\ldots,p$, D_i, F_i, E_i, M_i, and N_i are matrices of appropriate dimensions; D_i, F_i, and E_i denote the structure of the uncertainty; M_i and N_i are uncertain matrices (which may be scalars in special cases) subject to uncertainty bounds M_i and N_i (both symmetrical), respectively, as

$$M_iM_i^T\le\underline{M}_i;\qquad N_iN_i^T\le\underline{N}_i \tag{47}$$

The matrices $\underline{M}_i$ and $\underline{N}_i$ are chosen such that their product is an identity matrix (and the product is equal to one if the subject quantities are scalars). Although the same symbology is used, confusion need not arise between the D_i defined in Eq. (46) and the D_i defined in Eqs. (1) and (2) for the plant equations. The closed-loop system thus has structured uncertainty of the form

$$\Delta\tilde{A} = \begin{bmatrix} \Delta A & 0 \\ B_c \Delta C & 0 \end{bmatrix} = \sum_1^p \tilde{D}_i M_i N_i \tilde{E}_i \tag{48}$$

where

$$\tilde{D}_i \equiv \begin{bmatrix} D_i \\ B_c F_i \end{bmatrix}; \qquad \tilde{E}_i \equiv [E_i \;\; 0] \tag{49}$$

Again, one is not to confuse the uncertainty structure matrices $\tilde{D}_i$ and $\tilde{E}_i$, $i = 1,2,...,p$ with the noise parameter $\tilde{D}$ given in Eq. (9). In the robust control application, the design parameters D_1, D_2, $E_{1\infty}$, and $E_{2\infty}$ are chosen as in Eqs. (16) and (17). Therefore,

$$R_\infty = \begin{bmatrix} C^TC & 0 \\ 0 & 0 \end{bmatrix}; \quad \Delta R_\infty = \begin{bmatrix} \Delta C^TC + C^T\Delta C + \Delta C^T \Delta C & 0 \\ 0 & 0 \end{bmatrix} \tag{50}$$

Uncertainty Bounds

To find the controller A_c, B_c, and C_c by way of theorem 1, one must first find the uncertainty bound Ω that satisfies condition (30). The Petersen-Hollot bound[12,14] specified by

$$\Omega_P \equiv \sum_1^p (\tilde{D}_i \underline{M}_i \tilde{D}_i^T + Q\tilde{E}_i^T \underline{N}_i \tilde{E}_i Q) \tag{51}$$

bounds the linear terms of the perturbations, i.e.,

$$\Omega_P \le \Delta\tilde{A}Q + Q\,\Delta\tilde{A}^T \tag{52}$$

If we can find Ω_R that bounds the quadratic term, i.e.,

$$\Omega_R \le \gamma^{-2} Q\,\Delta R_\infty Q \tag{53}$$

then we may take

$$\Omega = \Omega_P + \Omega_R \tag{54}$$

Substituting ΔR_∞ of Eq. (50) into the right-hand side of Eq. (53) and invoking the definition of the spectral norm of matrices, one can readily see that an Ω_R may be chosen as

$$\Omega_R = \gamma^{-2} Q \begin{bmatrix} \bar{\sigma}(\Delta\bar{C})[2\bar{\sigma}(C) + \bar{\sigma}(\Delta\bar{C})]I & 0 \\ 0 & 0 \end{bmatrix} Q \tag{55}$$

which satisfies Eq. (53). The notation $\Delta\bar{C}$ means a matrix of which each element is the maximum modulus of the corresponding element of ΔC. Thus, $\Delta\bar{C}$ is a matrix of fixed elements.

Simple norm manipulations show that Ω_R may also be chosen as

$$\Omega_R = \gamma^{-2} Q \begin{bmatrix} C^TC + 2[\bar{\sigma}(\Delta\bar{C})]^2 I & 0 \\ 0 & 0 \end{bmatrix} Q \tag{56}$$

Loosely speaking, when ΔC is small relative to C, the quadratic bound of Eq. (55) is smaller than that of Eq. (56), and vice versa. In the following development, the quadratic bound of Eq. (55) will be used.

Surrogate System

Substitution of Eqs. (51), (54), and (55) into Eq. (29) gives

$$0 = \tilde{A}Q + Q\tilde{A}^T + \gamma^{-2} Q R'_\infty Q + V' \tag{57}$$

where, for the robust optimal system subject to Eqs. (16) and (17),

$$R'_\infty = \begin{bmatrix} R'_{1\infty} & 0 \\ 0 & 0 \end{bmatrix} \tag{58}$$

$$R'_{1\infty} = C^TC + \bar{\sigma}(\Delta\bar{C})[2\bar{\sigma}(C) + \bar{\sigma}(\Delta\bar{C})]I + \gamma^2 \sum_1^p E_i^T \underline{N}_i E_i \tag{59}$$

$$V' = V + \sum_1^p \tilde{D}_i \underline{M}_i \tilde{D}_i^T = \begin{bmatrix} V'_1 & V'_{12} B_c^T \\ B_c V'_{21} & B_c V'_2 B_c^T \end{bmatrix} \tag{60}$$

where

$$V'_1 = \sum_1^p D_i \underline{M}_i D_i^T \qquad V'_{12} = \sum_1^p D_i \underline{M}_i F_i^T \tag{61}$$

$$V'_{21} = \sum_1^p F_i \underline{M}_i D_i^T \qquad V'_2 = I + \sum_1^p F_i \underline{M}_i F_i^T \tag{62}$$

In applications where A and C are independent (which is quite common), the real-parameter uncertainty is modeled in such a way that, for each i, either D_i or F_i is zero and V' is diagonal. The additional weight required for a surrogate system with fixed parameters can now be determined by comparison of Eq. (57) with Eq. (23), with the aid of Eqs. (58–62). The actual system S'_Δ with design parameters chosen as in Eqs. (16) and (17) is drawn in Fig. 4. A surrogate system S'_0 is given in Fig. 5, where $R'_{1\infty}$ is given in Eq. (59), and V'_1 and V'_2 are given in Eqs. (61) and (62), respectively. The new output weighting in the surrogate system is $(R'_{1\infty})^{1/2}$. Note that the weight of the output is strengthened by a term involving an uncertainty bound of C and by the term $\Sigma E_i^T \underline{N}_i E_i$, which is associated with the common uncertainty structure of A and C. An additional noise w_1 with an intensity equal to an uncertainty bound $\Sigma D_i \underline{M}_i D_i^T$, which is particular to the uncertainty structure of $\bar{A}$, is added to the state integrator. The intensity of the Gaussian noise w_2 is strength-

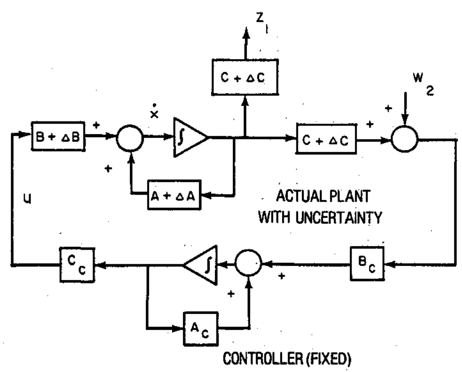

Fig. 4 Actual closed-loop system S'_Δ for robust control design.

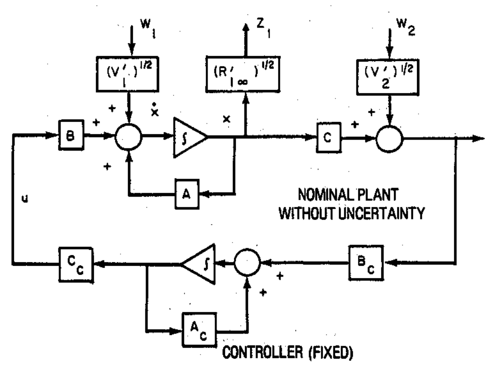

Fig. 5 Surrogate system S'_0.

ened by an amount equal to $\Sigma F_i \underline{M}_i F_i^T$, which is a bound particular to the uncertainty structure of C. In the case where A and C are mutually dependent, cross correlation between w_1 and w_2 must be considered.

It can be concluded from theorem 1 that to design a robust controller in the presence of both structured real-parameter uncertainty and unmodeled dynamics, such as the actual closed-loop system S'_Δ, one may model the real-parameter uncertainty as additional noise and additional weights in the existing noise inputs and measurement outputs, such as in the surrogate system S'_0, and carry out the H_∞/H_2 robust design based on lemma 1. The resulting controller will cause the actual system to have the same H_2 and H_∞ performance bounds as the surrogate system.

Robust Optimal Controller

The robust optimal control problem now is to synthesize A_c, B_c, and C_c for the surrogate system S'_0 such that Eq. (57) has a positive-definite solution Q while both trQR and γ are minimized. The positive definiteness of Q may be relaxed to non-negative definiteness. If so, the stability of the closed-loop system must be checked for all real-parameter variations, as noted at the end of the proof of lemma 2. The minimization of trQR can be carried out by first adjoining the right-hand side of Eq. (57) to trQR via a matrix Lagrange multiplier and then differentiating the resulting trace function with respect to Q, A_c, B_c, and C_c. A method for the exact minimization of γ is not known, but an approximate minimum of γ may be calculated by iterating the procedure of the minimization of Q while decreasing γ in each iteration until a positive-definite solution Q ceases to exist. The solution of Q, A_c, B_c, and C_c for the surrogate system S'_0 with a given γ may be found by following Bernstein and Haddad's procedure[15] for finding the controller parameters for a nominal system without real-parameter uncertainty. This is outlined as follows: first, adjoin the right-hand side of Eq. (57) to trQR via a symmetrical Lagrange multiplier matrix P to form

$$L = \mathrm{tr}\{QR + [\tilde{A}Q + Q\tilde{A}^T + \gamma^{-2}QR'_\infty Q + V']P\} \tag{63}$$

Next, partition Q and P into four blocks in accordance with the dimensions of the states of the plant and the controller. Now, differentiation of L with respect to Q (symmetric) yields three Riccati equations after an expansion with respect to the submatrices of Q. Expansion of Eq. (57) yields three more. Differentiation of L with respect to A_c, B_c, and C_c also yields three Riccati equations. Manipulation of these nine Riccati equations eventually yields implicit expressions for the submatrices of P and Q and explicit expressions of the controller parameters A_c, B_c, and C_c. The result is stated as follows.

Theorem 2: If (A_c, B_c, C_c) is given such that a positive-definite solution Q exists for Eq. (57), and trQR is minimal, then there exist non-negative-definite matrices Q' and P, and a positive-definite matrix $\tilde{Q}$, such that

$$C_c = -R_2^{-1}B^TP \tag{64}$$

$$B_c = (Q'C^T + V'_{12})(V'_2)^{-1} \tag{65}$$

$$A_c = A + BC_c - B_cC + \gamma^{-2}Q'R'_{1\infty} \tag{66}$$

$$Q = \begin{bmatrix} Q' + \tilde{Q} & \tilde{Q} \\ \tilde{Q} & \tilde{Q} \end{bmatrix} \tag{67}$$

where Q', $\tilde{Q}$, and P are solutions of

$$\begin{aligned} 0 = {} & [A - V'_{12}(V'_2)^{-1}C]Q' + Q'[A - V'_{12}(V'_2)^{-1}C]^T \\ & + Q'[\gamma^{-2}R'_{1\infty} - C^T(V'_2)^{-1}C]Q' \\ & + V'_1 - V'_{12}(V'_2)^{-1}V'_{21} \end{aligned} \tag{68}$$

$$\begin{aligned} 0 = {} & [A - BR_2^{-1}B^TP + \gamma^{-2}Q'R'_{1\infty}]\tilde{Q} \\ & + \tilde{Q}[A - BR_2^{-1}B^TP + \gamma^{-2}Q'R'_{1\infty}]^T \\ & + \gamma^{-2}\tilde{Q}R'_{1\infty}\tilde{Q} + (Q'C^T + V'_{12}) \\ & \times (V'_2)^{-1}(Q'C^T + V'_{12})^T \end{aligned} \tag{69}$$

$$\begin{aligned} 0 = {} & [A + \gamma^{-2}(Q' + \tilde{Q})R'_{1\infty}]^TP + P[A + \gamma^{-2}(Q' + \tilde{Q})R'_{1\infty}] \\ & + R_1 - PBR_2^{-1}B^TP \end{aligned} \tag{70}$$

Furthermore, the quadratic performance index bound is given by

$$\mathrm{tr}QR = \mathrm{tr}[(Q' + \tilde{Q})R_1 + \tilde{Q}PBR_2^{-1}B^TP] \tag{71}$$

Conversely, if there exist non-negative-definite matrices Q' and P and a positive-definite matrix $\tilde{Q}$ satisfying Eqs. (68–70), then A_c, B_c, C_c, and Q as given by Eqs. (64–67) satisfy Eq. (57), and Q is at least non-negative definite. The quadratic performance index is bounded by trQR. However, the minimality of trQR is not implied.

Proof: The differentiation of Eq. (63) with respect to A_c, B_c, C_c, and Q implies that A_c, B_c, C_c, and Q of Eqs. (64–67) satisfy Eq. (57) and the necessary conditions for minimal trQR. The bounding of the quadratic performance index follows from lemma 1. The definiteness of P, Q', and $\tilde{Q}$ has been established in the derivation procedure given in Ref. 15.

Note that for the special case where $V'_{12} = 0$ and $V'_{21} = 0$, Eqs. (64–71) have the same form as a special case under theorem 1 of Ref. 15.

Theorem 2 implies the asymptotic stability and the H_∞ and H_2 norm boundedness of the surrogate system S'_0 of Fig. 5. The following theorem states the asymptotic stability and H_∞ and H_2 norm boundedness of the actual system S'_Δ of Fig. 4 for all permissible parameter perturbations ΔA and ΔC.

Theorem 3: Let Q', $\tilde{Q}$, and P be a set of non-negative-definite or positive-definite solutions of Eqs. (68–70). Let the controller parameters A_c, B_c, C_c, and Q be given by Eqs. (64–67), respectively. If Q is positive definite, then the closed-loop system of Fig. 4, with $\Delta B = 0$ and ΔA and ΔC specified by Eqs. (46) and (47), has the following properties:

1) The closed-loop system S'_Δ is asymptotically stable for all permissible real-parameter uncertainties ΔA and ΔC.

2) $H(s, \Delta\tilde{A})$, the transfer function matrix from w_2 to z_1, satisfies the H_∞ norm constraint given in Eq. (12).

3) The quadratic performance index is constrained by Eq. (39), where the bound trQR satisfies the necessary condition of being minimal.

If Q of Eq. (67) is non-negative definite and the closed-loop system S'_Δ of Fig. 4 is asymptotically stable for all permissible ΔA and ΔC with $\Delta B = 0$, then conditions 2 and 3 are satisfied.

Proof: By virtue of theorem 2, Q is a positive-definite solution of Eq. (57). Since Eq. (57) is a special case of Eq. (29), the controller of Fig. 4, which is a special case of Fig. 1, satisfies the assumptions of lemma 2 when $\Delta B = 0$ and when ΔA and ΔC satisfy Eqs. (46) and (47). Therefore, theorem 1 implies theorem 3. The case of non-negative-definite Q follows from the argument noted after the proof of lemma 2.

Theorem 3 provides a method for synthesizing an optimal controller that is robust against both unmodeled dynamics and structured real-parameter uncertainties. A numerical example of its application is given in the next section.

Illustrative Example

To demonstrate the methodology described in this paper, a two-input/two-output system considered by Dickman[23] is studied. The plant for the system is given in terms of uncertain parameters a_i by

$$G(s,\Delta\tilde{A}) = \begin{bmatrix} \frac{1}{s-a_1} & \frac{1}{s+1} \\ \frac{1}{s+a_2} & \frac{1}{s+a_3} \end{bmatrix} \tag{72}$$

where

$$a_1 \in [0.6,\ 1.4]; \qquad a_2 \in [0.9,\ 1.1]; \qquad a_3 \in [1.8,\ 2.2] \tag{73}$$

The system must now be stabilized with the smallest possible H_∞ norm of the input-output transfer function given in Eq. (15).

A diagonal realization of the plant gives

$$A + \Delta A = \text{diag}[a_1 \ \ -a_2 \ \ -a_3 \ \ -1] \tag{74}$$

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}^T \qquad C = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \tag{75}$$

The nominal values of the elements of A are selected as the midpoints of the intervals of a_i. The matrices defining the structure of the parameter uncertainty are selected as

$$D_1 = [0.4 \ 0 \ 0 \ 0]^T; \qquad E_1 = [1 \ 0 \ 0 \ 0] \tag{76}$$

$$D_2 = [0 \ 0.1 \ 0 \ 0]^T; \qquad E_2 = [0 \ 1 \ 0 \ 0] \tag{77}$$

$$D_3 = [0 \ 0 \ 0.2 \ 0]^T; \qquad E_3 = [0 \ 0 \ 1 \ 0] \tag{78}$$

$$M_i, N_i \in [-1,\ 1]; \qquad \underline{M}_i = 110; \qquad \underline{N}_i = 1/\underline{M}_i \ \forall i \tag{79}$$

and D_1, D_2, $E_{1\infty}$, and $E_{2\infty}$ are selected as described in Eq. (16). The weights in the quadratic performance index of Eq. (13) are chosen to be

$$R_1 = R'_{1\infty} \qquad R_2 = 10^{-6} I \tag{80}$$

Computation of the controller parameters involves the solution of the coupled Riccati equations [Eqs. (68–70)]. Bernstein and Haddad[15] have discussed a homotopic continuation algorithm for solving this type of equation set. Basically, the procedure is to solve Eq. (70) for P using an initial guess of $\tilde{Q}$ and then to substitute this P into Eq. (69) to solve for $\tilde{Q}$. This process is then repeated until the solutions for P and $\tilde{Q}$ converge within a prespecified error. Note that Eq. (68) can be solved independent of Eqs. (69) and (70). Additionally, the initial guess of $\tilde{Q}$ affects the convergence rate of this iterative procedure. For large γ, however, the solution of Eq. (70) is nearly independent of the initial guess of $\tilde{Q}$, and the problem essentially reduces to an LQG design of the surrogate system. To take advantage of this fact, the algorithm should be first initiated with a reasonably large γ. Once a set of solutions P, Q', and $\tilde{Q}$ is obtained, the norm bound γ is decreased and the entire process repeated, until either a desired value of γ is achieved or no further decrease in its value is possible. The decreasing of γ is also a crucial step in the algorithm. The experience gained in this numerical exercise shows that, in order to maintain computational efficiency and provide a smooth transition to smaller values of γ, the initial guess of $\tilde{Q}$ for each new γ should be chosen as the solution $\tilde{Q}$ for the previous γ.

Little is known about how best to select the design parameters defining the structure of the uncertainty. The choice of D_i and E_i in Eqs. (76–78) is arbitrary, whereas M_i and N_i in Eq. (79) are chosen by ad hoc reasoning. Additionally, R is chosen to be as close to R'_∞ as possible [see Eq. (80)] because a special case then occurs where Eqs. (69) and (70) can be decoupled through a variable transformation.[15] As noted earlier, the question of how best to choose the design parameters or to affect the loop shape of the resulting system is a topic for further research.

The value of γ used in this exercise is 1.74 (4.811 dB). As a point of reference, the minimum γ achievable for the nominal system when no real parameter uncertainty is present (i.e., $a_1 = a_2 = 1$, $a_3 = 2$) is found to be 1.0497.

The transfer function of the controller achieving the value of $\gamma = 1.74$ is given by

$$K(s) = \frac{1}{D(s)} \begin{bmatrix} k_{11}(s) & k_{12}(s) \\ k_{21}(s) & k_{22}(s) \end{bmatrix} \tag{81}$$

where

$$D(s) = \left(1 + \frac{s}{2018.7}\right)\left(1 + \frac{s}{212.51}\right)\left(1 + \frac{s}{7.6342}\right)\left(1 + \frac{s}{3.0540}\right) \tag{82}$$

$$k_{11}(s) = -2.6735\left(1 + \frac{s}{1162.1}\right)\left(1 + \frac{s}{1.2048 + 0.13j}\right) \times \left(1 + \frac{s}{1.2048 - 0.13j}\right) \tag{83}$$

$$k_{12}(s) = -0.53452\left(1 + \frac{s}{2097.6}\right)\left(1 - \frac{s}{1.8173}\right) \times \left(1 + \frac{s}{1.3372}\right) \tag{84}$$

$$k_{21}(s) = 3.0833\left(1 + \frac{s}{1.5932 \times 10^4}\right)\left(1 + \frac{s}{2.0393}\right) \times \left(1 + \frac{s}{0.99997}\right) \tag{85}$$

$$k_{22}(s) = -0.23338\left(1 + \frac{s}{1948.0}\right)\left(1 + \frac{s}{1.0394}\right)\left(1 + \frac{s}{1.0000}\right) \tag{86}$$

For the reader's convenience, the state-space triple (A_c, B_c, C_c) for the case of $\gamma = 1.74$ is given by

$$A_c = \begin{bmatrix} -2352.5 & 1220.8 & 710.92 & -1612.3 \\ -2349.1 & 1219.5 & 710.67 & -1607.9 \\ 142.35 & -1039.8 & -925.02 & -182.93 \\ 142.35 & -1039.3 & -922.48 & -183.93 \end{bmatrix} \tag{87}$$

$$B_c = \begin{bmatrix} 6.6686 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.38200 & 0.81939 & 0.0 \end{bmatrix}^T \tag{88}$$

$$C_c = \begin{bmatrix} -2349.1 & 1220.8 & 710.92 & -1607.9 \\ 142.35 & -1039.3 & -922.48 & -182.93 \end{bmatrix} \tag{89}$$

Even though the matrices P, Q', and $\tilde{Q}$ turned out to be positive definite, the fact that this controller stabilizes the plant for all possible real-parameter uncertainties is verified by an extended version of Kharitonov's theorem.[21,22] The maximum singular value plot of the associated closed-loop transfer function is shown in Fig. 6, where all singular values are given in decibels. The nine curves correspond to values calculated for the nominal plant and for the eight vertices of the polytope of the parameter set (a_1, a_2, a_3). Note the insensitivity of the maximum singular value with respect to real-parameter perturbations. As noted in the Introduction, the design method proposed in this paper is conservative since the effects of real-parameter uncertainties are modeled by weighted external inputs. The authors are not aware of how to quantify the degree of conservatism theoretically. For this example, however, the closed-loop stability is preserved when the bounds on the real-parameter variations of Eq. (73) are increased by a factor of 4.11642.[24,25]

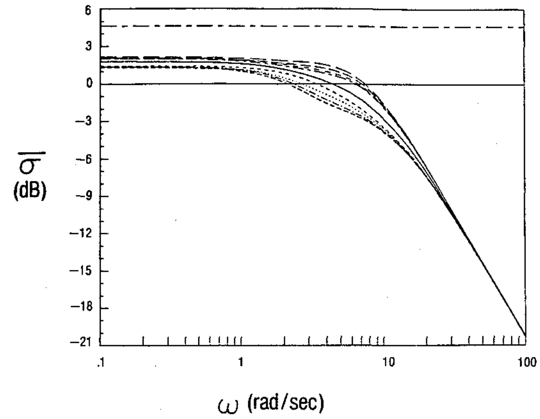

Fig. 6 Maximum singular value plot for the closed-loop system.

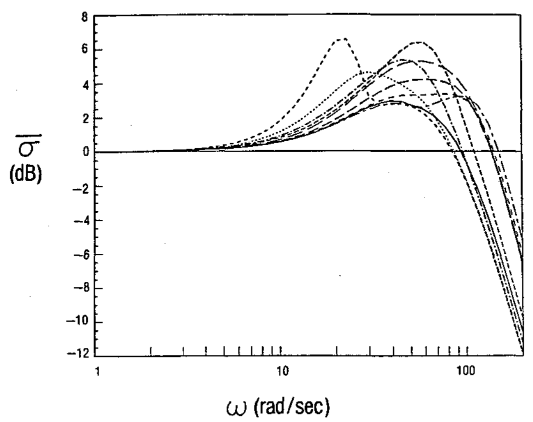

Fig. 7 Maximum singular value plot for the closed-loop system.

As a basis for comparison, the transfer function of the controller developed by Dickman[23] for this two-input/two-output uncertain systems is given in the form of Eq. (81) as

$$D(s) = s(1 + s/3)(1 + s/200)(1 + s/500) \tag{90}$$

$$k_{11}(s) = 466.7(1 + s)^2(1 + s/20) \tag{91}$$

$$k_{12}(s) = 466.7(1 - s)(1 + s)(1 + s/25) \tag{92}$$

$$k_{21}(s) = -933.3(1 + s)(1 + s/2)(1 + s/20) \tag{93}$$

$$k_{22}(s) = 466.7(1 + s)^2(1 + s/25) \tag{94}$$

Figure 7 shows the maximum singular values for the associated closed-loop transfer function using this controller. Again, the nine curves illustrated correspond to the nominal system and the eight vertices of the parameter set. In this case, the maximum singular value has no guaranteed upper bound, and it is more sensitive to real-parameter changes than that of the previous example.

Conclusions

This paper extends a Riccati equation approach and offers a design methodology for the synthesis of a robust controller accounting for both unmodeled dynamics (characterized as a single block uncertainty) and structured real-parameter uncertainty for multiple-input/multiple-output systems. The concept of a surrogate system provides a convenient means of robust synthesis subject to an invariant H_∞ bound in the presence of structured real-parameter uncertainty. In addition, the Riccati equation approach applied to achieve this goal further constrains the quadratic Gaussian performance index under a bound that satisfies a necessary condition of minimality.

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